CONTRACTOR REPORT BRL-CR-630

BRL

AN APPROXIMATION TO THE EIGENVALUES OF A LINEAR STABILITY PROBLEM FOR HIGH REYNOLDS NUMBER

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MAY 1990

U.S. ARMY LABORATORY COMMAND

BALLISTIC RESEARCH LABORATORY
ABERDEEN PROVING GROUND, MARYLAND

NOTICES

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY (Leave blank)

2. REPORT DATE
MAY 90

3. REPORT TYPE AND DATES COVERED
Contract, Nov 88 - Apr 90

4. TITLE AND SUBTITLE
An Approximation to the Eigenvalues of a Linear Stability Problem for High Reynolds Number

5. FUNDING NUMBERS
C: DAAL03-86-D-0001

6. AUTHOR(S)
Hall, Philip*

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
US Army Ballistic Research Laboratory
ATTN: SLCBR-DD-T
Aberdeen Proving Ground, MD 21005-5066

10. SPONSORING/MONITORING AGENCY REPORT NUMBER
BRL-CR-630

11. SUPPLEMENTARY NOTES
*University of Exeter, Exeter, England, and ICASE, Hampton, Virginia

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited

12b. DISTRIBUTION CODE

13. ABSTRACT (Maximum 200 words)
The spectrum of a sixth order differential operator is examined asymptotically in the high Reynolds number limit. The eigenvalue problem investigated arises in the study of the fluid motion in a coning and rotating fluid filled cylinder. It is shown that the approximation procedure derived at high Reynolds numbers predicts very accurately the required eigenvalues.

14. SUBJECT TERMS
Differential equations; eigenvalues, high Reynolds number, linear stability, Reynolds number; rotating liquid; stability, WKB method.

15. NUMBER OF PAGES
21

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	SAR -Unlimited

NSN 7540-01-280-5500

Standard Form 298 (Rev 2-89)
Prescribed by ANSI Std Z39-18
298-102

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ACKNOWLEDGEMENT

The author wishes to acknowledge the invaluable help that he received from Dr. Raymond Sedney and Mr. Nathan Gerber of the Ballistic Research Laboratory.

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1 Introduction and motivation

It is well-known that there can be a significant difference in behavior in flight between liquid-filled and solid-filled projectiles. The difference is caused by the motion of the liquid inside the spinning projectile. This motion causes forces to act on the projectile that can ultimately cause the flight of the projectile to be prematurely terminated by instability. The initial motion of the projectile necessarily causes the fluid motion in the cylinder to be time-dependent; later it can be assumed that the flow is steady. The destabilizing motion of the projectile induced by the forces exerted by the fluid is of small amplitude in its initial stages.

Several methods are available to find the fluid motion in a cylindrical container performing small amplitude oscillations; here we mention two of these approaches. First, a finite difference approach to the equations of motion can be used[1]. In this formulation, the Navier-Stokes equations are marched forward in time until an equilibrium state is reached. This is not a time-accurate method, however. The method is most economical at relatively small Reynolds numbers and prohibitively expensive[2] at Reynolds numbers of the order of 10,000.

Secondly, a spatial eigenvalue procedure has been developed by Hall, Sedney, and Gerber[3]. This procedure can be used to determine inexpensively the forced fluid motion at Reynolds numbers up to about 2000. The method produces results in excellent agreement with experimental observations and agrees with the finite difference results in most situations. A solution is obtained with this method by expressing the velocity and fluid pressure fields in terms of the eigenfunctions that describe the linear instability of solid-body rotation. However, because we are concerned with flows in cylinders of finite length, we can allow the axial wavenumber of the perturbations to be complex while the frequency of the perturbations is real. The eigenvalue problem associated with this type of perturbation has an infinite number of eigenvalues, which can be numbered in some way, for example, by counting the number of internal zeros of the eigenfunctions. Essentially it is found that eigenvalues can be naturally split into groups of three with the eigenvalues within each group now being ordered on the basis of the number of zeros of the eigenfunctions. The major computational task associated with the spatial eigenvalue approach is

to determine the eigenvalues. Most of the calculations performed by the present authors were carried out by using initial guesses for the eigenvalues from interpolation in a table of eigenvalues[4] obtained by extrapolation from the eigenvalues at a Reynolds number of 10. In order to perform calculations over a wider range of parameter values it was necessary to find an alternative (and quick) way of obtaining initial guesses for almost all of the eigenvalues. This report demonstrates the most useful such method, which is based on the WKB method.

In this report we explain the implementation of the WKB method to find the eigenvalue spectrum for solid-body rotation. The reader is referred to References 3 and 4 for discussions of the spatial eigenvalue approach. Here the approach is outlined for the completely filled cylinder problem; the extension of the approach to the 'inner-rod' or 'partially-filled' case is straightforward.

2 The spectrum at high Reynolds numbers

As mentioned above, the motivation for this work comes from our previous study of forced motions in finite rotating/nutating cylinders[3]. We first present a brief description of the physical problem that motivates the present study. Consider a cylinder of radius a and height $2c$, rotating about its axis with angular velocity Ω. The cylinder is filled with an incompressible fluid of density ρ and kinematic viscosity ν. After transient effects (spin-up) have died out, the fluid motion in the absence of gravity is solid body rotation. If the cylinder is made to cone at constant rate τ with respect to an inertial reference frame, the motion is a perturbation on solid body rotation. In coning motion, the axis of the cylinder moves on a cone; for convenience, the vertex of the cone is taken to be at the geometrical center of the cylinder. The angle between the cylinder and the cone axis is K_0. We define the Reynolds number Re by

$$Re = \frac{(\Omega + \tau \cos K_0)a^2}{\nu}$$

and the nondimensional perturbation frequency by

$$f = \frac{\tau}{\Omega + \tau \cos K_0}.$$

For the linearized problem, $\cos K_0 = 1$. Time, length, velocity and pressure are made nondimensional by $(\Omega+\tau)^{-1}$, a, $(\Omega+\tau)a$ and $\rho a^2(\Omega+\tau)^2$, respectively. Our starting point, therefore, is the linearized stability equations for a three-dimensional time-dependent disturbance to solid body rotation in a circular cylinder. We restrict our attention to the mode with azimuthal wavenumber unity and axial wavenumber k. This wavenumber is taken to be complex while the frequency of the disturbance is real; notice that at finite values of the Reynolds number, solid body rotation is stable so that no real eigenvalues k can exist . At infinite Reynolds numbers the flow is neutrally stable so that for high values of the Reynolds number we expect to find eigenvalues k with small imaginary part. For the inertial reference frame, the appropriate stability equations are:

$$
\begin{aligned}
[Re^{-1}(\Delta_1 - r^{-2}) - iM]u + \left(2 + \frac{2iRe^{-1}}{r^2}\right) v - p_r = 0 \\
[Re^{-1}(\Delta_1 - r^{-2}) - iM]v - \left(2 + \frac{2iRe^{-1}}{r^2}\right) u + \frac{ip}{r} = 0 \\
[Re^{-1}(\Delta_1) - iM]w + kp = 0 \\
(ru)_r - iv + krw = 0
\end{aligned}
\qquad (2.1a,b,c,d)
$$

Here r denotes the radial variable with the cylinder located at $r=1$ while u, v, w denote the radial, azimuthal, and axial perturbation velocities and p denotes the perturbation pressure. The operator Δ_1 is given by

$$
\Delta_1 \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \{k^2 + \frac{1}{r^2}\}, \qquad (2.2)
$$

and $M = f - 1$ with f equal to the perturbation frequency. If we are concerned with the case where there is no inner cylinder then (2.1) must be solved for $k = k(Re, f)$ such that

$$
\begin{aligned}
u - iv = w = p = 0, \quad r = 0 \\
u = v = w = 0, \quad r = 1
\end{aligned}
\qquad (2.3a,b)
$$

The first of these conditions ensures that the velocity and pressure fields of the disturbance are regular at the origin and the second condition is the no-slip condition for a viscous fluid. Notice that if we were investigating the 'inner rod' problem or the 'two-fluid 'problem, then the inner boundary conditions would be altered and applied instead at a finite value of r.

Our aim therefore is to determine the infinite spectrum of eigenvalues at a given value of f at high values of Re. From our previous experience (e.g., see Reference 3) we know that for positive values of the real part of k, there are three distinct branches of eigenvalues. Two of these branches, I and III, are in the first quadrant while a third branch, II, is in the fourth quadrant. We know that on any one of these branches the asymptotic expansion of any eigenvalue takes the form

$$k = k_0 + Re^{-\frac{1}{2}} k_1 + Re^{-1} k_2 + \cdots \tag{2.4a}$$

or the form

$$k = Re^{\frac{1}{2}} k_0 + Re^{-\frac{1}{2}} k_1 + \cdots . \tag{2.4b}$$

The eigenfunction associated with k_0 in (2.4a) can be expressed in terms of Bessel functions and k_1 depends on the mode number m (basically the number of zeros) of the eigenfunction. In fact, for large m it can be shown that $k_1 \sim m$, so that the first two terms in (2.4a) will formally break down when $m \sim Re^{\frac{1}{2}}$. In this situation the radial derivatives will formally be of size $Re^{\frac{1}{2}}$ so that the basic assumptions leading to (2.4a) no longer apply. It is this regime that we will now concentrate on since our previous experience has shown that (2.4a,b) are only useful in predicting accurate approximations to the first few eigenvalues on the different branches.

For modes with wavenumbers of size $0(Re^{\frac{1}{2}})$ the disturbance varies on a length scale $0(Re^{-\frac{1}{2}})$ over most of the flowfield; we anticipate this structure by seeking WKB type solutions of (2.1) with

$$\{u, v, w, p\} = \exp\{iRe^{\frac{1}{2}} \int^{r} \theta(\tilde{r})d\tilde{r}\} \sum_{n=0}^{\infty} (\underline{u}_n, \underline{v}_n, \underline{w}_n, Re^{\frac{-1}{2}} \underline{p}_n) Re^{-\frac{n}{2}}. \tag{2.5}$$

At this stage the viscous derivatives in the radial direction are of size $0(Re)$ while diffusion in the axial direction will be of size $0(k^2)$. Thus in order to balance the diffusion of vorticity in the radial and axial directions we must take $k = 0(Re^{\frac{1}{2}})$ so that (2.4a,b) must now be replaced by the expansion

$$k = \sqrt{Re}\, k_0 + k_1 + \frac{1}{\sqrt{Re}} k_2 + \cdots . \tag{2.6}$$

The WKB phase function θ appearing in (2.5) is determined by substitution for $\{u, v, w, p\}$ from (2.5) into (2.1) and retaining only the dominant terms in the limit $Re \to \infty$. We obtain

$$\begin{aligned} -[\theta^2 + k_0^2 + iM]\underline{u}_0 + 2\underline{v}_0 &= i\theta \underline{p}_0 \\ [\theta^2 + k_0^2 + iM]\underline{v}_0 + 2\underline{u}_0 &= 0 \\ -[\theta^2 + k_0^2 + iM]\underline{w}_0 + k_0 \underline{p}_0 &= 0 \\ i\theta \underline{u}_0 + k_0 \underline{w}_0 &= 0, \end{aligned} \tag{2.6b}$$

and these equations have a consistent solution if

$$(\theta^2 + k_0^2 + iM)^2(\theta^2 + k_0^2) + 4k_0^2 = 0, \tag{2.7}$$

so there are six possible values for θ for a given value of k. If the expansion (2.5) is to be valid over a $0(1)$ length in r then only real values of θ are acceptable. Thus we insist that (2.7) has two real roots, in which case it can be shown that the other four roots are complex. Without any loss of generality we can suppose that $\pm\theta_1$ are the two real roots of (2.7) while the remaining (complex) roots are denoted by $\pm\theta_2, \pm\theta_3$. Hence our expansion (2.5) must now be written down as a sum over the two acceptable real solutions of (2.7); alternatively these exponential solutions can be combined in terms of trigonometric functions.

At next order the linear equations to determine $(\underline{u}_1, \underline{v}_1, \underline{w}_1, \underline{p}_1)$ are obtained but are found to be forced by the zeroth order function $(\underline{u}_0, \underline{v}_0, \underline{w}_0, \underline{p}_0)$ and their derivatives with respect to the slow scale r. The consistency of these equations leads to a first order differential equation for $\underline{u}_0$ that can be solved to give

$$\underline{u}_0 = \frac{1}{r^{\frac{1}{2}}} \exp\{i\tilde{k}_1 r\} \tag{2.8}$$

in which the constant $\tilde{k}_1$ is given by

$$\tilde{k}_1 = \frac{-k_0 k_1}{\theta} \left\{ 1 + \frac{4}{(L^2 + 2L(\theta^2 + k_0^2))} \right\}. \tag{2.9}$$

The quantity L is defined by

$$L = \theta^2 + k_0^2 + iM. \tag{2.10}$$

A few comments are in order before completing our asymptotic solutions for k in the large wavenumber limit. First, we note that the phase function θ determined above is constant so that the 'fast' dependence of the perturbed flow could equivalently be expressed in terms of the variable $r^* = Re^{\frac{1}{2}}r$. Thus an alternative and possibly more instructive method to solve the asymptotic problem for k would be to use a multiple scale approach using the variables $Re^{\frac{1}{2}}r,\ r,\ Re^{\frac{-1}{2}}r,\ldots$. Secondly, we notice that the slow dependence of $\underline{u}_0$ (and hence that of $\underline{v}_0, \underline{w}_0$ and $\underline{p}_0$) on r takes the usual WKB form for a second order differential equation in that it is proportional to $\{\int^r \theta(\tilde{r})d\tilde{r}\}^{\frac{-1}{2}}$. The extra exponential factor is introduced because the differential system under consideration here is of sixth order. We now introduce the following nomenclature:

$$u_n = \underline{u}_n \ \exp(i\ Re^{1/2}\ \theta\ r),\quad n = 0, 1, 2, \ldots$$

with similar expressions for v_n, w_n, and p_n.

The solution u_0 is singular in the limit of $r \to 0$ so there must be an inner viscous boundary layer at the origin in order to smooth out the singularity. We have seen that the only acceptable roots of (2.7) are θ_1 and $-\theta_1$ since the complex roots will lead to exponential growth when r increases or decreases. Thus for $r = 0(1)$, the appropriate forms for u_0, v_0 and w_0 ,determined by (2.6b) and (2.8) are

$$\begin{aligned}
-u_0 &= \frac{L_1 \sin\{Re^{\frac{1}{2}}\theta_1 r + \tilde{k}_1 r\}}{2r^{\frac{1}{2}}} + \frac{NL_1 \cos\{Re^{\frac{1}{2}}\theta_1 r + \tilde{k}_1 r\}}{2r^{\frac{1}{2}}} \\
v_0 &= \frac{\sin\{Re^{\frac{1}{2}}\theta_1 r + \tilde{k}_1 r\}}{r^{\frac{1}{2}}} + \frac{N \cos\{Re^{\frac{1}{2}}\theta_1 r + \tilde{k}_1 r\}}{r^{\frac{1}{2}}} \qquad\qquad (2.11a,b,c) \\
-k_0 w_0 &= \frac{-\theta_1 L_1 \cos\{Re^{\frac{1}{2}}\theta_1 r + \tilde{k}_1 r\}}{2r^{\frac{1}{2}}} + \frac{L_1 N\theta_1 \sin\{Re^{\frac{1}{2}}\theta_1 r + \tilde{k}_1 r\}}{2r^{\frac{1}{2}}}
\end{aligned}$$

Here the constant $L_1 = \theta_1^2 + k_0^2 + iM$, while N is a constant to be found .

We can see from (2.11) that we cannot choose θ_1 and N, the two constants at our disposal, in order to satisfy the no-slip condition directly at $r = 1$. The remedy is to allow for a viscous boundary layer at $r = 1$ in which, in effect , the exponentially decaying WKB solution can be used to satisfy the condition. In fact, it is easiest to simply use a boundary layer variable

$$\xi = Re^{\frac{1}{2}}\{1 - r\}$$

and note from (2.11) that the $r = 0(1)$ core solution at $r \to 1$ can be expressed as

$$\begin{aligned} v_0 \to \sin\theta_1\xi\, &[N\sin(Re^{\frac{1}{2}}\theta_1 + \tilde{k}_1) - \cos(Re^{\frac{1}{2}}\theta_1 + \tilde{k}_1)] \\ + \cos\theta_1\xi\, &[\sin(Re^{\frac{1}{2}}\theta_1 + \tilde{k}_1) + N\cos(Re^{\frac{1}{2}}\theta_1 + \tilde{k}_1)] \end{aligned} \tag{2.12}$$

together with similar expressions for u_0, w_0.

In order to match with the above limiting form of the core solution, we must expand (u, v, w, p) in the $\xi = 0(1)$ wall layer in the form

$$(u, v, w, p) = (\tilde{u}_0(\xi), \tilde{v}_0(\xi), \tilde{w}_0(\xi), Re^{\frac{-1}{2}}\tilde{p}_0(\xi)). \tag{2.13}$$

If the above expansions are substituted into (2.1) and like powers of $Re^{-\frac{1}{2}}$ are equated we find that

$$\left\{\frac{d^2}{d\xi^2} - k_0^2 - iM\right\}\tilde{u}_0 + 2\tilde{v}_0 = -\tilde{p}_0' \tag{2.14a}$$

$$\left\{\frac{d^2}{d\xi^2} - k_0^2 - iM\right\}\tilde{v}_0 - 2\tilde{u}_0 = 0 \tag{2.14b}$$

$$\left\{\frac{d^2}{d\xi^2} - k_0^2 - iM\right\}\tilde{w}_0 = -k_0\tilde{p}_0 \tag{2.14c}$$

$$-\tilde{u}_0' + k_0\tilde{w}_0 = 0. \tag{2.14d}$$

If we eliminate $\tilde{u}_0, \tilde{w}_0, \tilde{p}_0$, we find that

$$\left[\left(\frac{d^2}{d\xi^2} - k_0^2 - iM\right)^2\left(\frac{d^2}{d\xi^2} - k_0^2\right) - 4k_0^2\right]\tilde{v}_0 = 0 \tag{2.15}$$

so that the solution for $\tilde{v}_0$ that satisfies the no-slip condition at $r = 1$ is

$$\tilde{v}_0 = \alpha[\sin\theta_1\xi + A\cos\theta_1\xi + Be^{-i\theta_2\xi} - (A + B)e^{-i\theta_3\xi}] \tag{2.16}$$

Here A, B, α are constants to be determined and θ_2, θ_3 are the roots of (2.7) that have negative imaginary parts. The functions $\tilde{u}_0, \tilde{w}_0, \tilde{p}_0$ can then be found from (2.14) and the application of the no-slip condition on $\tilde{u}_0, \tilde{w}_0$ shows that

$$\begin{aligned} B &= -\frac{iL_1\theta_1}{-L_3\theta_3 + L_2\theta_2 + \frac{L_3\theta_3(L_2 - L_3)}{L_1 - L_3}} \\ A &= \frac{B(L_3 - L_2)}{L_1 - L_3} \end{aligned} \tag{2.17a, b}$$

while the matching with the coreflow requires that

$$\alpha = N \ \sin(Re^{\frac{1}{2}}\theta_1 + \tilde{k}_1) - \cos(Re^{\frac{1}{2}}\theta_1 + \tilde{k}_1)$$

$$A\alpha = \sin(Re^{\frac{1}{2}}\theta_1 + \tilde{k}_1) + N \ \cos(Re^{\frac{1}{2}}\theta_1 + \tilde{k}_1).$$

Solving these equations for A yields

$$A = \frac{\tan(Re^{\frac{1}{2}}\theta_1 + \tilde{k}_1) + N}{N \ \tan(Re^{\frac{1}{2}}\theta_1 + \tilde{k}_1) - 1}. \tag{2.17c}$$

Since A is already known, it remains for us to determine the constant N and then (2.17c) will constitute an eigenvalue problem for k_0 and k_1. This constant can only be determined by matching the coreflow solution with the solution of the inner boundary layer problem.

2a Inner boundary-layer problem

Here, we define $\eta = Re^{\frac{1}{2}}r$ and write

$$\begin{aligned}
u &= Re^{\frac{1}{4}}\bar{u}_0(\eta) + Re^{-\frac{1}{4}}\bar{u}_1(\eta) + \ldots, \\
v &= Re^{\frac{1}{4}}\bar{v}_0(\eta) + Re^{-\frac{1}{4}}\bar{v}_1(\eta) + \ldots, \\
w &= Re^{\frac{1}{4}}\bar{w}_0(\eta) + Re^{-\frac{1}{4}}\bar{w}_1(\eta) + \ldots, \\
p &= Re^{-\frac{1}{4}}\bar{p}_0(\eta) + \ldots.
\end{aligned} \tag{2.18}$$

We have anticipated in (2.18) that the disturbance in the center layer is $Re^{\frac{1}{4}}$ larger than in the core; this increase in size is necessary because of the dependence of the core solution on $r^{-\frac{1}{2}}$. The expansions (2.18) are substituted into (2.1) and like powers of $Re^{-\frac{1}{2}}$ are equated. Surprisingly, we find that $(\bar{u}_0, \bar{v}_0, \bar{w}_0, \bar{p}_0)$ satisfy the full equations (2.1), but with r replaced by η, k by k_0, and $Re = 1$.

Hence, the solution cannot be completely determined analytically in the core so (2.1) must be solved numerically with $Re = 1$. However, for larger values of η we can show from (2.1) with $Re = 1$ and r replaced by η that a large η solution of (2.1) is

$$\bar{v}_0 \sim \frac{\sin\theta_1\eta}{\eta^{\frac{1}{2}}} + \frac{a\cos\theta_1\eta}{\eta^{\frac{1}{2}}} + \frac{be^{-i\theta_2\eta}}{\eta^{\frac{1}{2}}} + \frac{ce^{-i\theta_3\eta}}{\eta^{\frac{1}{2}}}, \tag{2.19}$$

together with similar expressions for $\bar{u}_0, \bar{w}_0$ and $\bar{p}_0$. Here the constants a, b and c can only be found by integrating (2.1) numerically. This was done by first calculating three

independent solutions of (2.1) in $(0, \eta_\infty)$ where η_∞ is some suitably large value of η. These three solutions are then multiplied by constants such that $(\bar{u}_0, \bar{v}_0, \bar{w}_0, \bar{p}_0, \bar{v}_0', \bar{w}_0')$ determined in this way is continuous at η_∞ with the same function determined asymptotically for large η. This can be achieved by a suitable choice of these three constants and a, b, c, in (2.19). Clearly, no iteration is necessary so (2.1) is solved numerically only three times in this procedure. The matching of the core and the center layer solutions then requires that $N = a$, so that (2.17c) yields

$$Re^{\frac{1}{2}}\theta_1 + \tilde{k}_1 = n\pi + \tan^{-1}\left\{\frac{N+A}{NA-1}\right\} \tag{2.20}$$

which determines k_0 and $\tilde{k}_1$. In fact, since $Re >> 1$, (2.20) reduces to

$$Re^{\frac{1}{2}}\theta_1 = n\pi, \quad \tilde{k}_1 = \tan^{-1}\left\{\frac{N+A}{NA-1}\right\}, \tag{2.21}$$

where n, an integer, is formally $0(Re^{\frac{1}{2}})$. We can take $\tilde{k}_1$ to be defined by the principal value of $\tan^{-1}[(N+A)/(NA-1)]$ since adding 2π to $\tilde{k}_1$ can easily be shown to be formally equivalent (when $Re >> 1$) to a different choice of n in the equation for θ_1.

We now see how the eigenvalues k_n are to be determined for $Re >> 1$. First, we choose a large integer n and define

$$\theta_1 = \frac{n\pi}{Re^{\frac{1}{2}}}.$$

Secondly, we solve (2.7) for the three possible values of k_0 in the half plane $R\{k_0\} > 0$. For each of these values of k_0, we solve (2.1) to find $a = N$, then k_1 is found from (2.9) and (2.21).

2b A comparison with exact eigenvalues

In Figure 1 we present plots of the imaginary part vs. real part of k for both the present approximations and the exact values for the case $Re = 1000, f = 0.1$. The two sets of data are seen to be in close agreement.

3 Some observations on the use of the method

1. It should be noted that the method is formally justified when the core variation of the disturbance is on the fast $Re^{\frac{1}{2}}$ lengthscale. In effect, this means that the eigenfunctions must have many zeros in the domain of interest. Notwithstanding this observation, in the

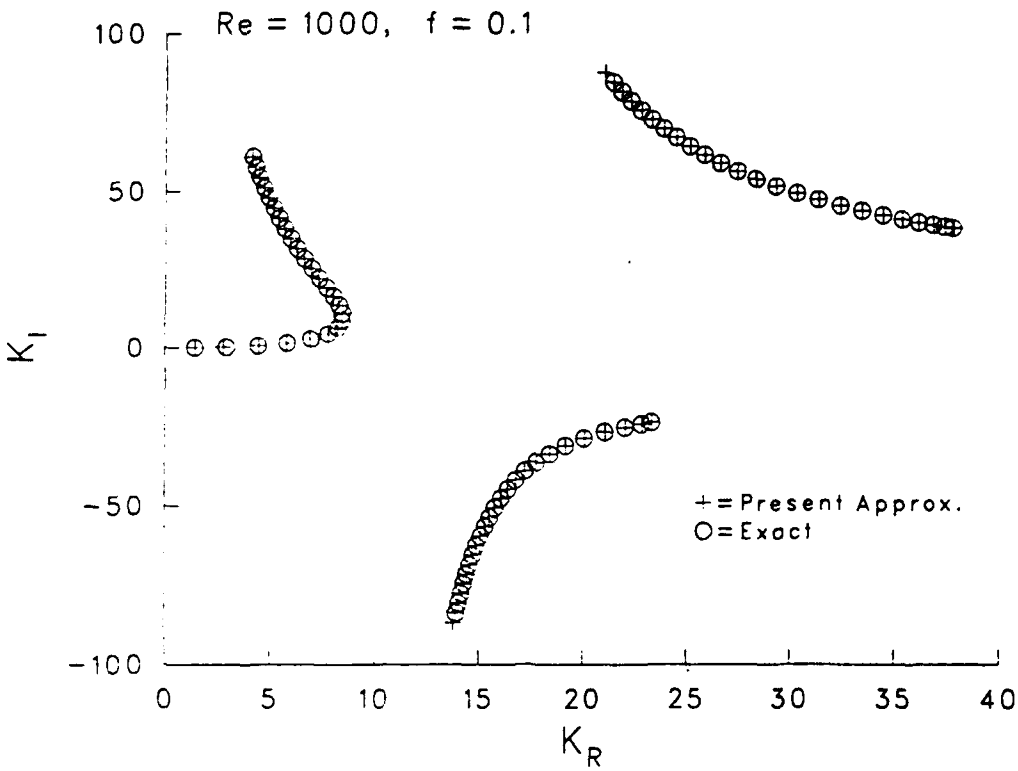

Figure 1. Plot of K_I vs. K_R for Re = 1000, f = 0.1 ($k \equiv K_R + i\,K_I$).

spirit of asymptotics we can now ignore this word of caution and apply the method for the case $n = 1, 2, 3, 4, 5, \ldots\ldots\ldots$ Though it is formally valid for the higher values of n, we can reasonably expect to get useful answers at small values too. However, since the formal requirement for the validity of the method is $n = 0(Re^{\frac{1}{2}})$, the method could well be perverse enough to give better guesses for the lower eigenvalues at the lowest values of Reynolds number used.

2. The order in which the eigenvalues come out of the WKB solution is related to the three branches we have discussed elsewhere. Essentially the eigenvalues fall into groups of three, one from each of the families. No sensible comment about the nature of the branch relationships for the first few eigenvalues predicted by the method can be made since inherently the method is being used out of its range of validity; all that one can say with any confidence or credibility is that if you go far enough down the series the eigenvalues predicted by the WKB method will fall out in groups of threes, one from each branch.

3. The method can be easily modified to predict eigenvalues in the presence of an inner rod. There are really just two cases worth considering. If the rod is of thickness $R\epsilon^{\frac{-1}{2}}$, the above formulation holds, but the inner equations now have to be solved subject to appropriate boundary conditions at a scaled value of r. Much more realistically, though, the usual situation will correspond to the case where the inner boundary is at a finite value of r. Suppose then that there is such a boundary at $r = q$. The boundary can be either a free surface or another rigid rod. In either case a boundary layer of thickness $Re^{\frac{-1}{2}}$ is now set up at the inner boundary in which the azimuthal velocity field will again satisfy (2.15) but with some appropriate variable replacing ξ. A solution corresponding to (2.16), but appropriate to the inner boundary condition, is then written down. Matching with the coreflow then gives the eigenrelation. Now, however, there is no inner problem to be solved numerically, basically because r is not small enough for the terms like $\frac{1}{r}$ to be comparable with derivatives with respect to r in the radial direction.

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